PLEASE NOTE ANSWERS TO SOME PROBLEMS ARE NOT UNIQUE!

WNE Linear Algebra Final Exam

Series B

8 February 2016

Please use separate sheets for different problems. Please provide the following data on each sheet

- **name, surname and your student number,**
- **number of your group,**
- **number of the corresponding problem and its series.**

Problem 1.
Let $V = \text{lin}((1,2,1,0),(0,2,1,1),(1,4,2,1),(3,8,4,1))$ be a subspace of $\mathbb{R}^4$.

a) find a system of linear equations which set of solutions is equal to V,
b) let $W_t = \{(x_1,x_2,x_3,x_4) \in \mathbb{R}^4 \mid 2x_1 + tx_2 + 2x_4 = 0\}$. For which $t \in \mathbb{R}$ the subspace V is a subset of W_t, i.e. $V \subset W_t$?

Solution.
Put vectors horizontally in a matrix and perform elementary row operations to get a reduced echelon form (up to column permutation).

$$\begin{bmatrix} 1 & 2 & 1 & 0 \\ 0 & 2 & 1 & 1 \\ 1 & 4 & 2 & 1 \\ 3 & 8 & 4 & 1 \end{bmatrix} \xrightarrow[r_4 - 3r_1]{r_3 - r_1} \begin{bmatrix} 1 & 2 & 1 & 0 \\ 0 & 2 & 1 & 1 \\ 0 & 2 & 1 & 1 \\ 0 & 2 & 1 & 1 \end{bmatrix} \longrightarrow \begin{bmatrix} 1 & 2 & 1 & 0 \\ 0 & 2 & 1 & 1 \end{bmatrix}$$

a) Any vector in the space V is equal to $x_1(1,2,1,0) + x_4(0,2,1,1) = (x_1, 2x_1 + 2x_4, x_1 + x_4, x_4)$ for some $x_1, x_4 \in \mathbb{R}$. This is a general solution of the following system of linear equations

$$\begin{cases} x_2 = 2x_1 + 2x_4 \\ x_3 = x_1 + x_4 \end{cases}$$

Answer:

$$\begin{cases} 2x_1 - x_2 + 2x_4 = 0 \\ x_1 - x_3 + x_4 = 0 \end{cases}$$

b) since $V = \text{lin}((1,2,1,0),(0,2,1,1))$ then $V \subset W_t$ if and only if $(1,2,1,0),(0,2,1,1) \in W_t$.

$$\begin{cases} 2 + 2t = 0 \\ 2t + 2 = 0 \end{cases}$$

Answer: for $t = -1$.

Problem 2.
Let $W \subset \mathbb{R}^4$ be a subspace given by the homogeneous system of linear equations

$$\begin{cases} x_1 + x_2 - 2x_3 + 2x_4 = 0 \\ 4x_1 + 5x_2 - 3x_3 + 4x_4 = 0 \end{cases}$$

a) find a basis and the dimension of the subspace W,
b) find a basis $\mathcal{A}$ of W such that the first two coordinates of the vector (1,-1,1,1) relative to $\mathcal{A}$ are $1, -1$.

Solution.
Solve the system of linear equations

$$\begin{bmatrix} 1 & 1 & -2 & 2 \\ 4 & 5 & -3 & 4 \end{bmatrix} \xrightarrow{r_2-4r_1} \begin{bmatrix} 1 & 1 & -2 & 2 \\ 0 & 1 & 5 & -4 \end{bmatrix} \xrightarrow{r_1-4r_2} \begin{bmatrix} 1 & 0 & -7 & 6 \\ 0 & 1 & 5 & -4 \end{bmatrix}$$

The general solution is

$$\begin{cases} x_1 = 7x_3 - 6x_4 \\ x_2 = -5x_3 + 4x_4 \end{cases}$$

that is $(7x_3 - 6x_4, -5x_3 + 4x_4, x_3, x_4) = x_3(7, -5, 1, 0) + x_4(-6, 4, 0, 1)$.

a) **Answer:** The basis of the set of solution is $(7, -5, 1, 0), (-6, 4, 0, 1)$. The dimension 2.
b) observe that $(1, -1, 1, 1) = (7, -5, 1, 0) + (-6, 4, 0, 1)$. Therefore $(1, -1, 1, 1) = (7, -5, 1, 0) - (6, -4, 0, -1)$
Answer: The basis is $(7, -5, 1, 0), (6, -4, 0, -1)$.

Problem 3.
Let $A = \begin{bmatrix} 1 & 2 \\ 1 & 0 \end{bmatrix}$

a) find matrix $C \in M(2 \times 2; \mathbb{R})$ such that $C^{-1}AC = \begin{bmatrix} s & 0 \\ 0 & -1 \end{bmatrix}$ for some $s \in \mathbb{R}$,
b) compute A^{100}.

Solution.
Compute the eigenvalues

$$\det \begin{bmatrix} 1-\lambda & 2 \\ 1 & -\lambda \end{bmatrix} = \lambda^2 - \lambda - 2 = (\lambda + 1)(\lambda - 2)$$

Therefore $\lambda = -1$ or $\lambda = 2$. Compute the eigenspaces

$$V_{(-1)}: \begin{bmatrix} 2 & 2 \\ 1 & 1 \end{bmatrix}$$

$V_{(-1)} = \{(x_1, x_2) \mid x_1 + x_2 = 0\} = \text{lin}((-1, 1))$

$$V_{(2)}: \begin{bmatrix} -1 & 2 \\ 1 & -2 \end{bmatrix}$$

$V_{(2)} = \{(x_1, x_2) \mid -x_1 + 2x_2 = 0\} = \text{lin}((2, 1))$

a) We see that $s = 2$. To get matrix C one need to put eigenvectors in columns in the corresponding order. **Answer:** $C = \begin{bmatrix} 2 & -1 \\ 1 & 1 \end{bmatrix}$.

b) let $D = \begin{bmatrix} 2 & 0 \\ 0 & -1 \end{bmatrix}$. Then $D = C^{-1}AC$ and therefore $A = CDC^{-1}$ so $A^{100} = CD^{100}C^{-1}$. Now $D^{100} = \begin{bmatrix} 2^{100} & 0 \\ 0 & (-1)^{100} \end{bmatrix}$ and from the formula $C^{-1} = \frac{1}{3}\begin{bmatrix} 1 & 1 \\ -1 & 2 \end{bmatrix}$.

$$A^{100} = \begin{bmatrix} 2 & -1 \\ 1 & 1 \end{bmatrix} \begin{bmatrix} 2^{100} & 0 \\ 0 & 1 \end{bmatrix} \frac{1}{3} \begin{bmatrix} 1 & 1 \\ -1 & 2 \end{bmatrix} = \frac{1}{3} \begin{bmatrix} 2^{101}+1 & 2^{101}-2 \\ 2^{100}-1 & 2^{100}+2 \end{bmatrix}$$

Answer: $A^{100} = \frac{1}{3}\begin{bmatrix} 2^{101}+1 & 2^{101}-2 \\ 2^{100}-1 & 2^{100}+2 \end{bmatrix}$

Problem 4.
Let $\mathcal{A} = ((1,-1,0),(0,2,1),(0,1,0))$ be an ordered basis of $\mathbb{R}^3$ and let $\mathcal{B} = ((0,1),(1,1))$ be an ordered basis of $\mathbb{R}^2$. The linear transformation $\psi\colon \mathbb{R}^3 \longrightarrow \mathbb{R}^2$ is given by the formula $\psi((x_1,x_2,x_3)) = (x_2+x_3, 2x_1-x_2)$. The linear transformation $\varphi\colon \mathbb{R}^2 \longrightarrow \mathbb{R}^2$ is given by the matrix $M(\varphi)_{\mathcal{B}}^{\mathcal{B}} = \begin{bmatrix} 1 & 0 \\ -1 & 1 \end{bmatrix}$.

a) find formula of φ,
b) compute matrix $M(\varphi \circ \psi)_{\mathcal{A}}^{\mathcal{B}}$.

Solution.a) By definition of a matrix of a linear transformation

$$\varphi((0,1)) = (0,1) - (1,1) = (-1,0),$$
$$\varphi((1,1)) = (1,1).$$

By linearity

$$\varphi((1,0)) = \varphi((1,1)) - \varphi((0,1)) = (1,1) - (-1,0) = (2,1)$$
$$\varphi((x_1,x_2)) = x_1\varphi((1,0)) + x_2\varphi((0,1))$$

Answer: $\varphi((x_1,x_2)) = (2x_1 - x_2, x_1)$

b)
$$M(\varphi \circ \psi)_{\mathcal{A}}^{\mathcal{B}} = M(\varphi)_{\mathcal{B}}^{\mathcal{B}} M(\psi)_{\mathcal{A}}^{\mathcal{B}}$$

To find $M(\psi)_{\mathcal{A}}^{\mathcal{B}}$ compute

$$\psi((1,-1,0)) = (-1,3) = 4(0,1) - (1,1)$$
$$\psi((0,2,1)) = (3,-2) = -5(0,1) + 3(1,1)$$
$$\psi((0,1,0)) = (1,-1) = -2(0,1) + (1,1)$$

Therefore

$$M(\psi)_{\mathcal{A}}^{\mathcal{B}} = \begin{bmatrix} 4 & -5 & -2 \\ -1 & 3 & 1 \end{bmatrix}$$

$$M(\varphi \circ \psi)_{\mathcal{A}}^{\mathcal{B}} = \begin{bmatrix} 1 & 0 \\ -1 & 1 \end{bmatrix} \begin{bmatrix} 4 & -5 & -2 \\ -1 & 3 & 1 \end{bmatrix} = \begin{bmatrix} 4 & -5 & -2 \\ -5 & 8 & 3 \end{bmatrix}$$

Answer: $M(\varphi \circ \psi)_{\mathcal{A}}^{\mathcal{B}} = \begin{bmatrix} 4 & -5 & -2 \\ -5 & 8 & 3 \end{bmatrix}$

Problem 5.
Let $V = \mathrm{lin}((1,1,0,1),(0,0,1,1),(1,1,1,2))$ be a subspace of $\mathbb{R}^4$.

a) find an orthonormal basis of V,
b) compute the orthogonal projection of $w = (0,2,0,1)$ on $V^{\perp}$.

Solution.
In order to find a basis of V put vectors horizontally in a matrix and perform elementary row operations to get a reduced echelon form

$$\begin{bmatrix} 1 & 1 & 0 & 1 \\ 0 & 0 & 1 & 1 \\ 1 & 1 & 1 & 2 \end{bmatrix} \xrightarrow{r_3-r_1} \begin{bmatrix} 1 & 1 & 0 & 1 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 1 & 1 \end{bmatrix} \longrightarrow \begin{bmatrix} 1 & 1 & 0 & 1 \\ 0 & 0 & 1 & 1 \end{bmatrix}$$

The basis is $(1,1,0,1),(0,0,1,1)$.

a) to get an orthogonal basis we need to run the Gram-Schmidt process on the basis $w_1 = (1,1,0,1), w_2 = (0,0,1,1)$.
$v_1 = w_1 = (1,1,0,1)$,
$v_2 = w_2 - \frac{w_2 \cdot v_1}{v_1 \cdot v_1} v_1 = (0,0,1,1) - \frac{1}{3}(1,1,0,1) = \frac{1}{3}(-1,-1,3,2)$. To get an orthonormal basis one need to normalize the vectors v_1, v_2.
Answer: Orthonormal basis of V is $\frac{1}{\sqrt{3}}(1,1,0,1), \frac{1}{\sqrt{15}}(-1,-1,3,2)$

b)

$$P_V(w) = \frac{w \cdot v_1}{v_1 \cdot v_1} v_1 + \frac{w \cdot v_2}{v_2 \cdot v_2} v_2$$

$$P_V(0,2,0,1) = \frac{(0,2,0,1)\cdot(1,1,0,1)}{(1,1,0,1)\cdot(1,1,0,1)}(1,1,0,1) + \frac{(0,2,0,1)\cdot(-1,-1,3,2)}{(-1,-1,3,2)\cdot(-1,-1,3,2)}(-1,-1,3,2) = (1,1,0,1)$$

$$P_{V^\perp}(w) = w - P_V(w)$$

$$P_{V^\perp}(0,2,0,1) = (0,2,0,1) - (1,1,0,1)$$

Answer: $P_{V^\perp}(0,2,0,1) = (-1,1,0,0)$

Problem 6.
Let

$$A = \begin{bmatrix} 1 & 2 & 1 & 2 \\ 0 & 2 & 2 & 2 \\ 1 & 3 & 1 & 4 \\ 2 & 4 & 2 & 5 \end{bmatrix}, \quad B_t = \begin{bmatrix} 3 & 1 & 0 & 5 \\ 2 & 1 & 2 & -3 \\ 0 & 0 & 1 & t \\ 0 & 0 & 1 & 4 \end{bmatrix},$$

where $t \in \mathbb{R}$.

a) compute $\det A$,
b) for which $t \in \mathbb{R}$ the matrix $B_t A^{-1}$ is invertible?

Solution.a)

$$\det A = \det \begin{bmatrix} 1 & 2 & 1 & 2 \\ 0 & 2 & 2 & 2 \\ 1 & 3 & 1 & 4 \\ 2 & 4 & 2 & 5 \end{bmatrix} \overset{r_4-2r_1}{=} \det \begin{bmatrix} 1 & 2 & 1 & 2 \\ 0 & 2 & 2 & 2 \\ 1 & 3 & 1 & 4 \\ 0 & 0 & 0 & 1 \end{bmatrix} = 1 \cdot (-1)^{4+4} \det \begin{bmatrix} 1 & 2 & 1 \\ 0 & 2 & 2 \\ 1 & 3 & 1 \end{bmatrix} \overset{r_3-r_1}{=}$$

$$= \det \begin{bmatrix} 1 & 2 & 1 \\ 0 & 2 & 2 \\ 0 & 1 & 0 \end{bmatrix} = 1 \cdot (-1)^{2+3} \det \begin{bmatrix} 1 & 1 \\ 0 & 2 \end{bmatrix} = -2$$

Answer: $\det A = -2$

b) matrix M is invertible if and only if $\det M \neq 0$

$$\det(B_t A^{-1}) = \det B \det(A^{-1}) = \det B_t (\det A)^{-1}$$

Therefore B_tA^{-1} is invertible if and only if $\det B_t \overset{\text{block matrix}}{=} \det\begin{bmatrix} 3 & 1 \\ 2 & 1 \end{bmatrix} \det\begin{bmatrix} 1 & t \\ 1 & 4 \end{bmatrix} = 4 - t$ is non-zero

Answer: for $t \neq 4$.

Problem 7.

Let $Q_t\colon \mathbb{R}^3 \longrightarrow \mathbb{R}$ be a quadratic form given by $Q_t((x_1, x_2, x_3)) = x_1^2 + 2x_2^2 + 2x_3^2 + 2x_1x_2 + 2tx_1x_3$.

a) for which $t \in \mathbb{R}$ the form Q_t is positive definite?

b) check if Q_t is either positive semidefinite or negative semidefinite for $t = 1$.

Solution.

Matrix of Q_t is $\begin{bmatrix} 1 & 1 & t \\ 1 & 2 & 0 \\ t & 0 & 2 \end{bmatrix}$

a) by Sylverster's criterion form Q_t is positive definite if and only if

$$\begin{cases} \det[1] = 1 > 0 \\ \det\begin{bmatrix} 1 & 1 \\ 1 & 2 \end{bmatrix} = 1 > 0 \\ \det\begin{bmatrix} 1 & 1 & t \\ 1 & 2 & 0 \\ t & 0 & 2 \end{bmatrix} = 2 - 2t^2 > 0 \end{cases}$$

Answer: $t \in (-1, 1)$

b) Substitute $t = 1$ and compute the eigenvalues.

$$\det\begin{bmatrix} 1-\lambda & 1 & 1 \\ 1 & 2-\lambda & 0 \\ 1 & 0 & 2-\lambda \end{bmatrix} = -\lambda(\lambda^2 - 5\lambda + 6) = -\lambda(\lambda - 2)(\lambda - 3)$$

All eigenvalues (i.e., $0, 2, 3$) are non-negative therefore the form Q_1 is positive semidefinite.

Problem 8.

Consider the following linear programming problem $6x_2 - x_3 + 3x_4 + x_5 \to \min$ in the standard form with constraints

$$\begin{cases} x_1 + 3x_2 + x_3 + 2x_4 = 6 \\ \quad\;\; + 2x_2 + x_3 + 2x_4 + x_5 = 2 \end{cases} \text{ and } x_i \geqslant 0 \text{ for } i = 1, \ldots, 5$$

a) which of the sets $\mathcal{B}_1 = \{3, 4\}, \mathcal{B}_2 = \{1, 5\}, \mathcal{B}_3 = \{3, 5\}$ are basic? Which of the sets $\mathcal{B}_1, \mathcal{B}_2, \mathcal{B}_3$ are basic feasible and which are basic infeasible?

b) solve the above linear programming problem using simplex method.

Solution.

The simplex tableau is

$$\left[\begin{array}{ccccc|c} 0 & 6 & -1 & 3 & 1 & 0 \\ \hline 1 & 3 & 1 & 2 & 0 & 6 \\ 0 & 2 & 1 & 2 & 1 & 2 \end{array}\right]$$

a)

$$\mathcal{B}_1\colon \left[\begin{array}{cc|c} 1 & 2 & 6 \\ 1 & 2 & 2 \end{array}\right] \text{ is not basic}$$

$$\mathcal{B}_2\colon \left[\begin{array}{cc|c} 1 & 0 & 6 \\ 0 & 1 & 2 \end{array}\right] \text{ is basic feasible}$$

$$\mathcal{B}_3\colon \left[\begin{array}{cc|c} 1 & 0 & 6 \\ 1 & 1 & 2 \end{array}\right] \xrightarrow{r_2 - r_1} \left[\begin{array}{cc|c} 1 & 0 & 6 \\ 0 & 1 & -4 \end{array}\right] \text{ is basic infeasible}$$

b) Start with $\mathcal{B} = \{1, 5\}$

$$\left[\begin{array}{ccccc|c} 0 & 6 & -1 & 3 & 1 & 0 \\ \hline 1 & 3 & 1 & 2 & 0 & 6 \\ 0 & 2 & 1 & 2 & 1 & 2 \end{array}\right] \xrightarrow{r_0 - r_2} \left[\begin{array}{ccccc|c} 0 & 4 & -2 & 1 & 0 & -2 \\ \hline 1 & 3 & 1 & 2 & 0 & 6 \\ 0 & 2 & 1 & 2 & 1 & 2 \end{array}\right]$$

The least number in the zeroth row is -2 (excluding the last column) which belongs to the third column so $s = 3$. The ratios between numbers in the last column and the third one are $\frac{6}{1}, \frac{2}{1}$ and the least ratio is 2 which corresponds to the second equation so $r = 2$. Therefore 3 enters the basic set and the second number leaves so $\mathcal{B} = \{1, 3\}$

$$\left[\begin{array}{ccccc|c} 0 & 4 & -2 & 1 & 0 & -2 \\ \hline 1 & 3 & 1 & 2 & 0 & 6 \\ 0 & 2 & 1 & 2 & 1 & 2 \end{array}\right] \xrightarrow[r_1 - r_2]{r_0 + 2r_2} \left[\begin{array}{ccccc|c} 0 & 8 & 0 & 5 & 1 & 2 \\ \hline 1 & 1 & 0 & 0 & -1 & 4 \\ 0 & 2 & 1 & 2 & 1 & 2 \end{array}\right]$$

All numbers in the zeroth row (excluding the last column) are positive therefore the basic set $\mathcal{B} = \{1, 3\}$ gives an optimal solution.
Answer: the minimum of the objective function is -2 and it is attained at the point $\overline{x}_{\{1,3\}} = (4, 0, 2, 0, 0)$ of the feasible set.